Solving Simultaneous Linear Equations by Gauss and Gauss-Jordan Elimination

Stuff You Should Already Know and Keep In Mind

You can add two or more equations without changing the values of the variables, and the order in which you add them doesn't matter. The values that solve each of the equations below are $x = 7$ and $y = -2$.

Equation 1	$x + 2y = 3$
Equation 2	$2x + 3y = 8$
Equation 3	$-5x + 15y = -65$

If we add those equations together, we get

$$-2x + 20y = -54$$

If we plug in $x = 7$ and $y = -2$ into that result, we get

$$-2(7)+ 20(-2) = -54, \text{ or}$$

$$-54 = -54$$

Subtracting one equation from another won't change the variables, either. Subtract equation 1 from equation 2.

$$\begin{array}{lrcr} & 2x + 3y & = & 8 \\ & -x - 2y & = & -3 \\ \text{To get} & x + y & = & 5 \end{array}$$

Replace x by 7 and y by -2 on the left side, to get $7 - 2 = 5$. So the left side and the right side are the same, and subtraction hasn't changed the values of the variables.

Multiplying or dividing an equation by a constant doesn't change the values of the variables, either. Just don't try to divide by zero.

Elimination Without Matrices

Before we get to matrices, here's how to do elimination with ordinary x's, y's, and z's. The elimination part is the heart of the matter. Matrices are just icing on the cake. They make it easier.
I'll start with a set of three equations in three unknowns, and tell you before we work the problem, that the values that satisfy all three equations are, $x = 1$, $y = -1$ and $z = 2$. Here are the equations.

$3x + 2y + z = 3$ row 1
$2x + y + 3z = 7$ row 2
$x + y + z = 2$ row 3

Ideally, in our first equation, the x-coefficient will be 1. In this case, it happens that row 3 starts that way. Which row is where doesn't affect the results, so let's swap rows one and three to get:

$x + y + z = 2$ new row 1
$2x + y + 3z = 7$ row 2
$3x + 2y + z = 3$ new row 3

Notice that I changed the row numbers. That doesn't hurt anything either, just makes it easier to stay oriented.

Now for the strategy. We want to eliminate the x from row 2 and the x and y from row 3. Then we divide row 3 by the coefficient of z. Once we do that, row 3 becomes the answer for z. Next, we back-substitute that z-value into row 2, which by this time contains only y and z. This lets us solve for y. Then we substitute both y and z values into row 1 to solve for x. That's the strategy. It may be hard to swallow in one gulp, but it will become clear as we actually do it.

Now let's start solving the things by eliminating that 2x from row 2.

Multiply row 1 by 2 and subtract the result from row 2.

$$\begin{aligned} 2x + y + 3z &= 7 \\ -2x - 2y - 2z &= -4 \end{aligned}$$

$$0x \quad -y \quad +z = 3$$

Replace row 2 with $-y + z = 3$ Your equations now look like this.

$x + y + z = 2$	row 1
$-y + z = 3$	row2
$3x + 2y + z = 3$	row3

Notice that row 1 hasn't changed. Only row 2 has changed. The new equation there is wholly different from the one in the old row 2, but if you were to plug in the values of y and z that I've told you solve all the equations we started with, you'll find they solve this one, too, AND we've eliminated x altogether, which was the idea.

Next we eliminate that 3x from row three. Multiply row 1 by three and subtract the result from row three.

$$\begin{aligned} 3x + 2y + z &= 3 \\ -3x - 3y - 3z &= -6 \end{aligned}$$

$$0x \quad -y \quad -2z = -3$$

This is your new row 3, and your set of equations look like this.

$x + y + z = 2$	row 1
$-y + z = 3$	row 2
$-y - 2z = -3$	row 3

Now we want to eliminate that -y in row three. Just subtract row two from row 3

$-y - 2z = -3$	Note: Subtracting $-y + z = 3$ is the same as adding
$+y - z = -3$	$y - z = -3$.

$$-3z = -6$$

Our equations are now

$x + y + z = 2$ row 1
$-y + z = 3$ row 2
$-3z = -6$ row 3

Next, divide row three by -3 to get $z = 2$. Hey, that's an answer! And we got it by elimination.

Our set of equations now looks like this.

$x + y + z = 2$ row 1
$-y + z = 3$ row 2
$z = 2$ row 3

Now we back-substitute, replacing the z in row 2 with 2.

$-y + z = 3$
$-y + 2 = 3$
$-y = 1$
$y = -1$

Another answer. Two down, one to go.

Finally, to get x, we substitute our values of z and y into the equation of row 1.

$x + y + z = 2$
$x - 1 + 2 = 2$
$x + 1 = 2$
$x = 1$

Now we've got them all. $x = 1$, $y = -1$, $z = 2$. And we got them by eliminating x from row two and x and y from row 3. We'll do the same thing in Gauss elimination using matrices, and it will be easier because we won't have to write x, y, or z, or whatever the variables may be, all over the place. But first we need to know a little about matrices and how they work.

Matrices

There are lots of things one can do with matrices, but I'll just cover the points needed to solve simultaneous linear equations using Gauss and Gauss-Jordan elimination.

A matrix consists of rows and columns of numbers. A square matrix which is what we'll use here, has an equal number of rows and columns, which are filled by numbers. A 3 X 3 matrix looks like the figures below, the first emphasizing the three rows, and the second emphasizing the three columns. But both are exactly the same matrix. Check the position of each number. They're identical.

$$\begin{array}{cl} \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{bmatrix} & \begin{array}{l} \text{row 1} \\ \text{row 2} \\ \text{row 3} \end{array} \end{array}$$

$$\begin{array}{ccc} \text{col 1} & \text{col 2} & \text{col 3} \end{array}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \\ 9 & 10 & 11 \end{bmatrix}$$

We identify the position of a number in a matrix by its row and column. The rows are numbered 1, 2, 3, ... from top to bottom, and the columns 1, 2, 3, ... from left to right. The number 10 in the matrix above is in row 3, column 2, and the number 3 is in row 1, column 3.

Excuse the hand-drawn brackets. My software doesn't do big brackets. Once we get into actual problems, I'll just leave the brackets to your imagination. Computer printouts of matrices do this all the time.

The rows and columns in our matrix will be occupied by the coefficients of the variables in our equations. For simplicity, I'll use 3 X 3 matrices, with variables x, y, and z, though you can have as many columns and matrices as you have variables in your equations, and in practice, you just assign the columns to match the variables in those equations. Don't forget, to get a unique solution, you need at least as many equations as variables.

Let's start with our old friends, first in equation form, then in matrix form.

$$3x + 2y + z = 3$$
$$2x + y + 3z = 7$$
$$x + y + z = 2.$$

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$$

These are just the coefficients of the left sides of the equations. This is a 3 X 3 matrix I'll call A.

What happened to the 3, 7, and 2 on the right side of the equal signs? Well, they're constants, with no variables attached. We'll get to them in another matrix. All we want here is the coefficients of the variables, that is, the coefficients from the left sides of the equations.

Now we need a 1 X 3 matrix for the constants on the right side. I'll call this one B. Each row of B will contain the constant that says what the left side of the equation in that row is equal to. B looks like this.

$$B = \begin{bmatrix} 3 \\ 7 \\ 2 \end{bmatrix}$$

Each equation's left side equals the corresponding constant on the right side. We can put the two matrices together into an augmented matrix. Matrix B augments matrix A, as is shown in the following figure.

$$C = \left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 3 & 7 \\ 1 & 1 & 1 & 2 \end{array}\right]$$

This is the augmented matrix, which I'll call C. The first row, 3 2 1 | 3, is the augmented matrix's way of saying $3x + 2y + z = 3$.

With that verbiage out of the way, let's start actually solving a set of three equations in three unknowns using Gauss Elimination.

Gauss Elimination

We'll start with the same three equations we just did without matrices. Here are the equations again, and below them is the augmented matrix.

$$3x + 2y + z = 3$$
$$2x + y + 3z = 7$$
$$x + y + z = 2$$

$$\left[\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 3 & 7 \\ 1 & 1 & 1 & 2 \end{array}\right]$$

And here's what it will look like after we've finished our Gauss elimination.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Note that the bottom row says $z = 2$. We'll get y by substituting z into the equation of row 2, and x by substituting y and z into the equation of row 1.

Notice that in the augmented matrix above, the left side has a diagonal of ones and a triangle of zeros below it. The diagonal and the triangle of zeros beneath it are always there in the finished product.

So much for preliminaries. Now let's actually solve these equations using our matrices.

$$\begin{array}{ccc|c} 3 & 2 & 1 & 3 \\ 2 & 1 & 3 & 7 \\ 1 & 1 & 1 & 2 \end{array}$$

To avoid unnecessary fractions, we'd rather start with a 1 as the first coefficient of x. So let's swap the first and third rows. This gives us

$$\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 1 & 3 & 7 \\ 3 & 2 & 1 & 3 \end{array}$$

Get used to looking for an advantageous swap. We'll do a lot of it.

Now we need to manipulate rows 2 and 3 so that the first entry in each of these rows is zero. This is what we mean by elimination. We're eliminating coefficients.

Let's start by working on row 2, which starts with 2. To do this, we multiply row 1 by 2 and subtract the result from row 2.

2(1 1 1 2) = 2 2 2 4 This is equivalent to $2(x + y + z) = 2x + 2y + 2z = 4$

Row 2	2	1	3	7	
Subtract from it	2	2	2	4	
To get	0	-1	1	3	This is our new row 2, which means $0x - y + z = 3$.

We leave rows one and three as they were, and the augmented matrix now looks like this.

$$\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & 1 & 3 \\ 3 & 2 & 1 & 3 \end{array}$$

Next, we eliminate the 3 in column 1 of row 3. Again, we multiply row 1 by something. This time we want to get rid of a 3, so we multiply row 1 by 3, to get 3 3 3 6, which we subtract from row 3.

Row 3 is	3	2	1	3
Subtract	3	3	3	6
To get	0	-1	-2	-3

That would be our new row 3, but just for convenience, let's multiply it by -1 to get 0 1 2 3, and call that our new row 3 instead.

Now our augmented matrix looks like this.

$$\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & -1 & 1 & 3 \\ 0 & 1 & 2 & 3 \end{array}$$

Next we have to eliminate the 1 in column 2 of row 3. Row 1, which we've used before, won't help us because it has a 1 in column 1, which would replace the zero we just got rid of in row 3. But our solution is a snap. Since the first non-zero in row 2 is minus one, and the first non-zero in row 3 is 1, we just add the two rows to eliminate that one.

Add	0	1	2	3
And	0	-1	1	3

To get 0 0 3 6 for row 3.

The matrix now looks like this

$$\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 6 \end{array}$$

Finally, let's clean it up and divide row3 by 3 to get the matrix below.

$$\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{array}$$

Read that third row. The z column is 1 and the augmentation column is 2. What does that tell us? By golly, we've solved the first part of the problem. $z = 2$

Now we can get y by substituting z into the equation given in row 2.

$y + 2z = 3$
$y + 4 = 3$
$y = -1$

Finally, we substitute $y = -1$ and $z = 2$ into the equation of row1 to solve for x

$x + y + z = 2$

$x + (-1) + 2 = 2$
$x + 1 = 2$
$x = 1$

So now we've got the solutions for all three variables. $x = 1$, $y = -1$, $z = 2$. We're finished with the Gauss Elimination method for this problem.

Gauss-Jordan Elimination

More often than not, Gauss Elimination with back-substitution is the easiest way to get a solution. But you might actually need Gauss-Jordan someday, or more likely, your instructor might insist on you going all the way with Gauss-Jordan, or you might have to deal with a computer program that does Gauss-Jordan by default. If nothing else, the Jordan result is elegant. What's more, you can go online, punch in your original matrix, and in a second or two, get the Gauss-Jordan results to verify your work.

Starting with the finished Gauss matrix, all we have to do now is get zeros in every spot above the diagonal of our matrix A. When we're finished, we'll be able to read all three results directly from the augmented matrix.

We start here with the finished Gauss Matrix.

$$\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{array}$$

Now all we have to do is eliminate the ones in columns 2 and 3 of row 1, and the 2 in the second row, leaving us with just the diagonal of ones on the left side, our much-revised A-matrix.

To eliminate the second one in row 1, subtract row 2 from it.

$$\begin{array}{cccc} 1 & 1 & 1 & 2 \\ 0 & -1 & -2 & -3 \\ & & & \\ 1 & 0 & -1 & -1 \end{array}$$

Now our matrix is

$$\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{array}$$

To eliminate the -1 remaining in row 1, add row 3 to row 1

$$\begin{array}{cccc} 1 & 0 & -1 & -1 \\ 0 & 0 & 1 & 2 \\ & & & \\ 1 & 0 & 0 & 1 \end{array}$$

We're finished with row 1, and the matrix becomes

$$\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{array}$$

Notice that row 1 now gives us the value of x.

All that's left to do now is to eliminate that 2 in row 2. To do this, multiply row 3 by 2 and subtract the result from row 2.

2 times row 3 is 0 0 2 4

Now the final subtraction

0 1 2 3
0 0 -2 -4

0 1 0 -1

Our finished product looks like this.

1 0 0 | 1
0 1 0 | -1
0 0 1 | 2

From that, you can read your answers directly from the augmented matrix. The first row says x = 1. The second row says y = -1, and the third row says z = 2

Nothing to it but a lot of tedious number-crunching. Sounds like work for a computer, doesn't it? And that's exactly what it is. On the Internet you can find sites that allow you to type in your augmented matrix, press the "solve" button, and get your result in Gauss-Jordan form in a second or two. It's easier than doing it by hand, especially if you're dealing with many equations and an equal number of unknowns.

Now that you know how it's done, I'll toss in a few more examples with answers. Practice certainly doesn't make perfect, but it does make problem solving a lot easier.

Example 1: The variable values are, x = 1, y = 3, z = -2.

$4x + 7y + 2z = 21$
$x + 2z = -3$
$3x + 2y = 9$

4 7 2 21
1 0 2 -3
3 2 0 9

Notice that the second row has a zero in the y-column because there's no y in that equation. The third row has a zero for a missing z.

The first thing to do with this one is move the second row to the top position to avoid fractions where possible, and bring the bottom equation to second position and the top one to the bottom. It's not obvious, but moving row 3 to the row 2 position saves us from more fractions. This done, the augmented matrix is now the figure below.

```
1 0 2  | -3
3 2 0  |  9
4 7 2  | 21
```

Multiply row 1 by 3 and subtract row 2 from it.

```
 3  0 6   -9
-3 -2 0   -9

 0 -2 6  -18        New row 2
```

Divide new row 2 by -2 to get

```
0 1 -3   9
```

Multiply row 1 by 4 and subtract row 3 from it.

```
 4   0  8   -12
-4  -7 -2   -21
```

0 -7 6 -33 Multiply this by -1 just to decrease number of minus signs.

0 7 -6 33 New Third row. The matrix is now as shown below.

```
1 0  2 |  -3
0 1 -3 |   9
0 7 -6 |  33
```

Multiply row 2 by 7 and subtract row 3 to get

0 7 -21 63
0 -7 6 -33

0 0 -15 30

Divide row 3 by -15 to get

0 0 1 -2

New Matrix

1 0 2 | -3
0 1 -3 | 9
0 0 1 | -2

Row 3 tells us z = -2

Back-substituting into row 2, we find

y - 3z = 9
y -3(-2) = 9
y + 6 = 9
y = 3

Now we substitute y = 3 and z = -2 into the equation of the first row
x +0 +2(-2) = -3 or
x -4 = -3 or
x= 1

Using Gauss elimination and back-substitution, we've found that x = 1, y = 3, z = -2 solves all three equations.

Now on to Gauss-Jordan.

1 0 2 | -3
0 1 -3 | 9
0 0 1 | -2

Multiply row 3 by 2 and subtract the result from row 1

1 0 2 -3
0 0 -2 4

1 0 0 1

This tells us x = 1, which we read from our new matrix below

1 0 0 | 1
0 1 -3 | 9
0 0 1 | -2

Now to eliminate the -3 from row 2. Multiply row 3 by 3 and subtract the result from row 2

0 -1 3 -9
0 0 -3 6

0 -1 0 -3 Multiply by -1 to get

0 1 0 3 This is your new row 2. Now we can write the final matrix.

1 0 0 | 1
0 1 0 | 3
0 0 1 | -2

We've completed the Gauss-Jordan elimination and can write the answer for each variable by inspection. Row one: x = 1, row two, y = 3 row three, z = -2.

Example 2.

This one turns out to be a pussycat, though it's not obvious when you first look at the equations. The solution is $x = 3$, $y = 1$, $z = 9$.

$8x + 3y + 5z = 72$
$2x + 2y = 8$
$y + 2z = 19$

Right off the bat, notice that equation 2 is divisible by 2. Divide it to get $x + y = 4$. Also, swap it with row 1 to get a 1 in the upper left hand corner of your matrix.

Here's the matrix we'll use.

$$\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 8 & 3 & 5 & 72 \\ 0 & 1 & 2 & 19 \end{array}$$

Multiply row 1 by 8 and subtract the result from row 2.

$$\begin{array}{cccc} 8 & 3 & 5 & 72 \\ -8 & -8 & 0 & -32 \\ & & & \\ 0 & -5 & 5 & 40 \end{array}$$

This could become new row 2, but notice that each term is divisible by 5. Divide by 5, and the new row 2 becomes

$0\ -1\ \ 1\ \ 8$

Your matrix now looks like this

$$\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & -1 & 1 & 8 \\ 0 & 1 & 2 & 19 \end{array}$$

Now just add rows 2 and 3 to get your new row 3.

$$\begin{matrix} 0 & -1 & 1 & 8 \\ 0 & 1 & 2 & 19 \\ \\ 0 & 0 & 3 & 27 \end{matrix}$$

Notice that both terms of the new row 3 are divisible by three. Do the division and get 0 0 1 9 for row 3.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & -1 & 1 & 8 \\ 0 & 0 & 1 & 9 \end{array}\right]$$

We see that z = 9. From here it's just a matter of back-substituting to get values for y and x.

Now for Gauss-Jordan. We start with the finished Gauss matrix and eliminate the non-zeros in the triangle above the diagonal of ones.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 4 \\ 0 & -1 & 1 & 8 \\ 0 & 0 & 1 & 9 \end{array}\right]$$

First we eliminate the second 1 in row 1 by adding rows one and two. It gives us a new row 1 of 1 0 1 12 for the matrix below.

$$\begin{matrix} 1 & 0 & 1 & 12 \\ 0 & -1 & 1 & 8 \\ 0 & 0 & 1 & 9 \end{matrix}$$

Rats! That introduces another 1 into row 1. Get rid of it by subtracting row 3 from row 1 to get you final first row of 1 0 0 3.

We're just about done, and the matrix looks like this.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & -1 & 1 & 8 \\ 0 & 0 & 1 & 9 \end{array}\right]$$

Multiply row 2 by -1 to get

0 1 -1 -8 Now add row 3 to get

0 1 0 1 for the final second row.

The matrix below is the finished Gauss-Jordan elimination

$$\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 9 \end{array}$$

All done. $x = 3$, $y = 1$, and $z = 9$

Example 3.

This one's just a little harder because it's more dense, and because you'll encounter fractions. It's not more difficult in principle, just more tedious. The values $x = 2$, $y = 3$, $z = -1$ satisfy the equations below.

$$x + 2y - 5z = 13$$
$$3x - y - 3z = 6$$
$$2x + y + z = 6$$

The equations give you the augmented matrix

$$\begin{array}{ccc|c} 1 & 2 & -5 & 13 \\ 3 & -1 & -3 & 6 \\ 2 & 1 & 1 & 6 \end{array}$$

Same old drill as last time. Multiply row 1 by 3 and subtract the result from row 2. That gives you 0 -7 12 -33. Multiply that by -1 to get 0 7 -12 33, which becomes your new row 2.

For row 3, multiply row 1 by 2 and subtract the result from row 3. That gives you 0 -3 11 -20. Multiply by minus one to get rid of a couple of the minus signs. That gives 0 3 -11 20 for the new row 3. The matrix now becomes:

1 2 -5 | 13
0 7 -12 | 33
0 3 -11 | 20

Now we're stuck with fractions.

Multiply row 2 by 3/7 to get 0 7(3/7) -36/7 + 99/7, or 0 3 -36/7 99/7, which we add to row 3.

0 3 -36/7 99/7
0 -3 77/7 -140/7

0 0 41/7 -41/7

Now multiply by 7/41 to get the third row 0 0 1 -1

Our matrix becomes

1 2 -5 | 13
0 -7 12 | -33
0 0 1 | -1

Row 3 tells us that z = -1
Substituting z = -1 into row 2 gives

-7y +12z = -33
-7y -12 = -21
-7y = -21
y=3

Substituting the values for y and z into row 1 gives

x + 2y -5z = 13
x + 2(3) -5(-1) = 13
x + 6 + 5 = 13
x + 11 = 13
x =2

There you have it. x =2, y = 3, and z = -1

If you want to go on to the Gauss-Jordan finale, you'll get

1	0	0	2
0	1	0	3
0	0	1	-1

I leave the details to you. I don't feel like working out another one of these things, and it's full of fractions. You'll know you've got it right when you get the augmented matrix shown above.

Sets of Equations That Won't Work

This set of equations doesn't look evil, but it is, as you'll see when you try to solve it. Well, maybe it's not really evil, but we can't get unique solutions for x, y, and z with it.

$x + 2y + 5z = 6$
$2x + 3y + 8z = 7$
$x + y + 3z = 1$

1	2	5	6
2	3	8	7
1	1	3	1

Multiply row 1 by 2 and subtract it from row 2

2	3	8	7
-2	-4	-10	-12
0	-1	-2	-5

Multiply the result by -1 to get : 0 1 2 5

Your matrix becomes

$$\begin{array}{ccc|c} 1 & 2 & 5 & 6 \\ 0 & 1 & 2 & 5 \\ 1 & 1 & 3 & 1 \end{array}$$

Subtract row 3 from row 1

$$\begin{array}{cccc} 1 & 2 & 5 & 6 \\ -1 & -1 & -3 & -1 \\ \\ 0 & 1 & 2 & 5 \end{array}$$

Your matrix becomes

$$\begin{array}{ccc|c} 1 & 2 & 5 & 6 \\ 0 & 1 & 2 & 5 \\ 0 & 1 & 2 & 5 \end{array}$$

Now you see the problem. Two rows are identical. No need for both of them. So subtract row3 from row 2, and your matrix becomes

$$\begin{array}{ccc|c} 1 & 2 & 5 & 6 \\ 0 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 \end{array}$$

The last row is all zeros and utterly useless. We're left with two equations and three unknowns. That won't give us unique solutions for x, y, and z. We're left with

$$\begin{aligned} x +2y +5z &= 6, \text{ and} \\ y +2z &= 5 \end{aligned}$$

What can we do? Well, not as much as we'd like. We can solve for y or x in terms of z. $y = 5 - 2z$ or $x = -z - 4$. Now if we assign a value to z, we can get solutions for x and y. If we arbitrarily say $z = 1$, then $x = -5$ and $y = 3$. Plug those values into our original equations, and you'll find they work just fine. The trouble is that if we assign z a different number, we get different solutions for x and y, and they work in our equations, too.

Not only don't we have a unique set of values of x, y, and z that satisfy our equations, we have an infinite number of solutions.

A more obvious way we can get into this situation is if two of our three equations are equivalent. Look at the matrix below.

$$\begin{array}{ccc|c} 2 & 5 & 1 & 6 \\ 1 & 1 & 2 & 3 \\ 12 & 30 & 6 & 36 \end{array}$$

At first glance it might look okay, but notice that each number in equation three is exactly six times the value of the corresponding number in equation 1, which means the two equations are the same. Divide equation 3 by 6 and suddenly rows one and three become identical.

$$\begin{array}{ccc|c} 2 & 5 & 1 & 6 \\ 1 & 1 & 2 & 3 \\ 2 & 5 & 1 & 6 \end{array}$$

Again, you've got only two equations and three unknowns, and as in the previous example, an infinite number of solutions. An infinite number of sets of x, y, and z values that solve the original equations. The best you can do is solve for one variable in terms of the others.

More Examples

I'm pretty sick of typing these things, but you need more examples to practice with. So here's how to make your own and get a better feel for the whole process at the same time. Start with the answer, for example, $x = 5$, $y= -3$, and $z = 2$. Then write down the left sides of three linear equations of your own choice and substitute the values you've chosen for x, y, and z into each of them to find the right half of each equation. You might also try one with four equations and four unknowns to see how fast the number of operations grows as matrix size increases. Anyway, here's and example of making your own examples.

$x = 5, y = -3, z = 2$

$x + 2y + 3z =$
$2x - y + 7z =$
$x + 6y - 9z =$

Plug the values of your variables into the first equation and you get $5 - 6 + 6 = 5$.
So the first equation is $x + 2y + 3z = 5$.
The second one works out to $2x - y + 7z = 27$.
The third is $x + 6y - 9z = -31$

Your augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 5 \\ 2 & -1 & 7 & 27 \\ 1 & 6 & -9 & -31 \end{array}\right]$$

Have at it. You'll know if you got it right because you started with the answers. Incidentally, remember you don't have to put all three variables into every equation. So practice leaving a variable out here and there. $2x + z = 12$ would be perfectly okay for one equation. And try sets of equations with more variables, and also sets of equations that don't have unique solutions.

Finally, when in doubt, go onto the Internet and google "Gauss Jordan calculator" without the quotes. That should pop up assorted calculators. My own favorite is Greg Thatcher's. You just type in the numbers of each row and press "solve." You'll have your answers in a second or two, complete with the steps the computer took to get the solution. By the way, the computer's not put off by fractions; it just plows on through them. You probably don't want to do it exactly the computer's way.

A final note: You know I hate fractions, and there's no instructor to make me work problems with fractions in them. But you're probably not that lucky. When you make up examples, be sure to make some of them with fractions. Don't go overboard, though. Fractions are time consuming, and you probably won't need them too much for class work unless your instructor is a sadist. Here a couple of tough ones with simple answers you can try if you really want to.

$2x/3 - 9y + 28/5z = 491/15$
$4x - 3y + 4z = 27$
$2x - y + 8z = 37$

I'll give you a clue. The answer is pretty simple, but the path to it is not.

Or you could try

$2x/3 - 9y + 28/5 = -41/15$
$4x - 3y + 4z = 5$
$2x - y + 8 = 9$

You should get $x = 1$, $y = 1$, $z = 1$ for this one, and the fractions involved are nasty. But it's good practice.

Maybe it's more realistic to try something like

$x/3 + y + z = 4/3$
$x + y + 2z = 1$
$5x + 2y + 3z = 6$

This one's fairly easy as fractions go, and you should get
$x = 1$, $y = 2$, $z = -1$

At this point I leave you to practice, and maybe even enjoy your linear equations as you get good at them.